ASYMPTOTIC SERIES FOR SOME PAINLEVÉ VI SOLUTIONS

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Introduction.

The study of asymptotic (as independent variable tends to a singular point) properties of Painlevé transcendents is one of the most important fields in modern theory of integrable nonlinear ODE's. The Painlevé equations are known to be integrable in the sense of commutative matrix representation (Lax pairs). One has six matrix equations

$$D_z L_j - D_x A_j + [L_j, A_j] = 0, \; j = 1, 2, ..., 6, \tag{1}$$

where $D_x = d/dx$; $L_j = L_j(y, y', x, z)$, $A_j = A_j(y, y', x, z)$ are 2*2 matrices that rationally depend on spectral parameter z, and the j-th Painlevé equation $y'' - P_j(y, y', x) = 0$ is equivalent to (1). The matrices L_j, A_j were written in paper [1].

The goal of this paper is to analyze asymptotic behavior of the sixth Painlevé transcendent using the so-called Whitham method. The PVI case is tedious due to large amount of calculations, so it is easier to illustrate the basic ideas of the method (which are the same for all the six equations) on technically the simplest case of PI.

The matrices L_1 and A_1 look as follows:

$$L_1 = \begin{pmatrix} 0 & 1 \\ y - z & 0 \end{pmatrix}, \; A_1 = \begin{pmatrix} -y' & 2y + 4z \\ -x - y^2 + 2yz - 4z^2 & y' \end{pmatrix}. \tag{2}$$

Introduce now new variable X and replace all the variables x explicitly entering formula (2) by X : $L_j = L_j(y, y', X, z)$, $A_j = A_j(y, y', X, z)$. For such matrices we have the following Lemma.

Lemma 1. Let ϵ be some real positive number. Then system

$$\epsilon D_z L_j - D_x A_j + [L_j, A_j] = 0 \tag{3}$$

is equivalent to system

$$D_x X = \epsilon, \quad y'' = P_j(y, y', X) = 0. \tag{4}$$

Proof can be obtained via direct computation. So, for PI the system (4) has the form

$$D_x X = \epsilon, \quad y'' - 3y^2 - X = 0.$$

Calculations for the other Painlevé equations are principally analogous and can be extracted from paper [2].

Lemma 2. Solution of equation (3) as $\epsilon = 0$ and $X = const$ can be represented by the following formula:

$$y_0(x) = f_j(\tau + \Phi; \vec{a}), \quad j = 1, 2, ..., 6, \tag{5}$$

where $\tau = xU,\ U = U(\vec{a});\ f_j$ are periodic functions which can be explicitly written out in terms of Weierstrass or Jacobi elliptic functions for any of the six Painlevé equations. The vector $\vec{a}(X)$ consists of parameters that detrmine the elliptic function f_j. Φ is some phase shift.

The proof uses the latter equation of system (4) where X is put to constant value. In the case of the first Painlevé function f_1 is Weierstrass $\wp-$function:

$$f_1 = 2\wp\left(x + \Phi; g_2, g_3\right); \ g_2 = -X, \ g_3 = -F_1/4, \tag{6}$$

where F_1 is some parameter. the formula (6) was first figured out in paper [3].

Now admit that number ϵ ispositive and small. We look for solutions to equation (3) in the form of formal series in parameter ϵ :

$$y(x) = y_0(x) + \epsilon y_1(x) + ..., \tag{7}$$

so that parameters determining the elliptic function $y_0 = f_j$ obey some special nonlinear ODE usually called Whitham equation or modulation equation. Thus, we look for the main term of series (7) in the form

$$y_0(\tau, X) = f_j\left(\epsilon^{-1}S(X) + \Phi(X); \vec{a}(X)\right), \quad D_X S = U.$$

Lemma 3. The Whitham equation can be written in the following form:

$$D_X \det A_j = \overline{a_{22}D_z l_{11}} + \overline{a_{11}D_z l_{22}} - \overline{a_{12}D_z l_{21}} - \overline{a_{21}D_z l_{12}}, \tag{8}$$

where $A_j = (a_{mn})\,,\ L_j = (l_{mn})\,,\ m,n = 1,2$, the bar means averaging over period of the elliptic function (5).

Proof. One can easily see that equation (3) as $\epsilon = 0$ indicates independence for spectral characteristics of matrix A_j of variable x. So the condition $D_x \det A_j = 0$ holds. Formal introduction of variable X induces the change for differentiation rule: $D_x \to UD_\tau + \epsilon D_X$, where parameter ϵ is put to be small and positive. Further the condition (3) yields equation

$$a'_{n,m} = \epsilon D_z l_{n,m} + [L_j, A_j]_{n,m}\,,\ n,m = 1,2.$$

Substituting this into equality

$$D_x \det A_j = a'_{11}a_{22} + a'_{22}a_{11} - a'_{12}a_{21} - a'_{21}a_{12},$$

we change the differentiating rule and obtain the following:

$$(UD_\tau + \epsilon D_X) \det A_j = \epsilon\,(a_{22}D_z l_{11} + a_{11}D_z l_{22} - a_{12}D_z l_{21} - a_{21}D_z l_{12}) + O\left(\epsilon^2\right).$$

Now average, i.e. integrate over the period (in "fast" variable τ). The averaging kills complete derivatives in τ which gives the claim.

Corollary 1. There exists unique coefficient of the polynomial $\det A_j$ with non-trivial dynamics in X in force of the modulation equation. Denote this coefficient F_j. Thus the Whitham system can be written as unique ODE on F_j.

The corollary can be verified via direct calculations for all the six equations. For PI we have the following:

$$\det A_1 = 16z^3 + 4Xz - F_1,$$

where $F_1 = (y')^2 - 2y^3 - 2yX$. The modulation equation (8) takes the form

$$D_X \det A_1 = 4z + 2\overline{y}$$

and can be rewritten as $D_X F_1 = -2\overline{y}$. Taking into account the solution (6), we obtain:

$$D_X F_1 = -2\eta/\omega = 2e_1 + 2(e_3 - e_1)E/K,$$

where $E = E(k),\ K = K(k)$ are complete elliptic integrals:

$$K = \int_0^1 \frac{dz}{\sqrt{(1-z^2)\,(1-k^2z^2)}},\ E = \int_0^1 \sqrt{\frac{1-k^2z^2}{1-z^2}}dz,\ k^2 = \frac{e_2 - e_3}{e_1 - e_3},$$

$e_{1,2,3}$ are roots of Weierstrass polynomial $R_3(t) = 4t^3 - g_2 t - g_3$; $g_2 = -X$, $g_3 = -F_1/4$.

Corollary 2.The simplest way for obtaining the elliptic ansatz f_j is solving equations

$$F_j = const_1, \; X = const_2. \tag{9}$$

Lemma 4. The elliptic ansatz (5) forms the main term y_0 in series in small parameter ϵ (7) for solution to system (3).

To prove this one should see that perturbation of solution to system (3) with $\epsilon = 0$ runs continuously while ϵ obtains small non-zero value. The appropriate elementary calculations are illustrated here on the simplest example of PI. So, for $\epsilon > 0$ system (3) is $y'' = 3y^2 + X_0 + \epsilon x$, where X_0 is constant. Via simple manipulations this can be reduced to condition

$$2dx = \frac{dy}{\sqrt{2y^3 + X_0 + const}} + O(\epsilon)$$

which means that the main term of the series (7) is the function f_1 (see (6)) on condition that x does not belong to small neighborhoods of singularities of the elliptic function f_1.

Now we can prove the following theorem.

Theorem 1. The function y_0 determined by formulas (5) and (8) forms the main term of asymptotic series for solution of appropriate Painlevé equation as $|x|$ tends to infinity.

Proof. The scale transformation $x \to \epsilon x$ leads to change $D_x X = \epsilon \mapsto D_x X = 1$ in formula (4). Therefore the expansion (7) in small parameter ϵ turns to series in negative powers of large variable x.

2. PVI and the Whitham method.

The sixth (and the most common) Painlevé equation

$$y'' = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x}\right)(y')^2 - \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x}\right)y' + \tag{10}$$

$$\frac{y(y-1)(y-x)}{x^2(x-1)^2}\left(\alpha + \beta\frac{x}{y^2} + \gamma\frac{x-1}{(y-1)^2} + \delta\frac{x(x-1)}{(y-x)^2}\right),$$

where the Greek letters denote free parameters, can be obtained as the compatibility condition of the following linear system of equations:

$$D_z Y = A_6(z,x)Y(z,x), \quad D_x Y = L_6(z,x)Y(z,x), \tag{11}$$

where

$$A_6(z,x) = \begin{pmatrix} a_{11}(z,x) & a_{12}(z,x) \\ a_{21}(z,x) & a_{22}(z,x) \end{pmatrix} = \frac{A^0}{z} + \frac{A^1}{z-1} + \frac{A^x}{z-x},$$

$$A^i = \begin{pmatrix} u_i + \theta_i & -\omega_i u_i \\ \omega_i^{-1}(u_i + \theta_i) & -u_i \end{pmatrix}, \ i = 0, 1, x, \ L_6(z,x) = -A^i \frac{1}{z-x}. \tag{12}$$

Put

$$A^\infty = -\left(A^0 + A^1 + A^x\right) = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix},$$

$$k_1 + k_2 = -(\theta_0 + \theta_1 + \theta_x), \quad k_1 - k_2 = \theta_\infty,$$

$$a_{12}(z) = -\frac{\omega_0 u_0}{z} - \frac{\omega_1 u_1}{z-1} - \frac{\omega_x u_x}{z-x} = \frac{k(z-y)}{z(z-1)(z-x)},$$

$$u = a_{11}(y) = \frac{u_0 + \theta_0}{y} + \frac{u_1 + \theta_1}{y-1} + \frac{u_x + \theta_x}{y-x}, \tag{13}$$

$$\widehat{u} = -a_{22}(y) = u - \frac{\theta_0}{y} - \frac{\theta_1}{y-1} - \frac{\theta_x}{y-x}.$$

Then $u_0 + u_1 + u_x = k_2, \ \omega_0 u_0 + \omega_1 u_1 + \omega_x u_x = 0,$

$$\frac{u_0 + \theta_0}{\omega_0} + \frac{u_1 + \theta_1}{\omega_1} + \frac{u_x + \theta_x}{\omega_x} = 0, \ (x+1)\omega_0 u_0 + x\omega_1 u_1 + \omega_x u_x = k, \ x\omega_0 u_0 = k(x)y,$$

which are solved as

$$\omega_0 = \frac{ky}{xu_0}, \ \omega_1 = -\frac{k(y-1)}{(x-1)u_1}, \ \omega_x = \frac{k(y-x)}{x(x-1)u_x},$$

$$u_0 = \frac{y}{x\theta_\infty} S_1,$$

where

$$S_1 = y(y-1)(y-x)\widehat{u}^2 + \left[\theta_1(y-x) + x\theta_x(y-1) - 2k_2(y-1)(y-x)\right]\widehat{u} +$$

$$k_2^2(y-x-1) - k_2\left(\theta_1 + x\theta_x\right),$$

$$u_1 = -\frac{y-1}{(x-1)\theta_\infty} S_1,$$

where

$$S_1 = y(y-1)(y-x)\widehat{u}^2 + \left[(\theta_1+\theta_\infty)(y-x) + x\theta_x(y-1) - 2k_2(y-1)(y-x)\right]\widehat{u} +$$

$$k_2^2(y-x) - k_2\left(\theta_1 + x\theta_x\right) - k_1k_2, \tag{14}$$

$$u_x = \frac{y-x}{x(x-1)\theta_\infty} S_\infty,$$

where

$$S_\infty = y(y-1)(y-x)\widehat{u}^2 + \left[\theta_1(y-x) + x(\theta_x+\theta_\infty)(y-1) - 2k_2(y-1)(y-x)\right]\widehat{u} +$$

$$k_2^2(y-1) - k_2\left(\theta_1 + x\theta_x\right) - xk_1k_2.$$

The compatibility condition for (11) implies

$$y' = \frac{y(y-1)(y-x)}{x(x-1)}\left(2u - \frac{\theta_0}{y} - \frac{\theta_1}{y-1} - \frac{\theta_x - 1}{y-x}\right). \tag{15}$$

Thus y satisfies PVI with the parameters

$$\alpha = \frac{1}{2}\left(\theta_\infty - 1\right)^2,\ \beta = -\frac{1}{2}\theta_0^2,\ \gamma = \frac{1}{2}\theta_1^2,\ \delta = \frac{1}{2}\left(1 - \theta_x^2\right).$$

Now we apply ideas described in the previous paragraph to asymptotic analysis of the sixth Painlevé transcendent. First calculate determinant for the matrix A_6. Using formulas (13), (14) one obtains:

$$a_{11}(z) = -R^{-1}(z)S,$$

where

$$S = k_1z^2 + z\left[x\left(u_0 + u_1 + \theta_0 + \theta_1\right) + u_0 + \theta_0 + u_x + \theta_x\right] + O\left(z^0\right),$$

$$a_{22}(z) = -R^{-1}(z)\left\{k_2z^2 - z\left[(x+1)u_0 + xu_1 + u_x\right] + O\left(z^0\right)\right\},$$

where $R(t) = t(t-1)(t-x);\ O\left(z^j\right)$ means powers of z of order not higher than j. The entries $a_{12},\ a_{21}$ yield terms of lower order in z, so they can be ignored

while computing the two higher terms of polynomial $\det A_6$. Therefore we have the following:

$$\det A_6 = R^{-2}(z)S,$$

where

$$S = k_1 k_2 z^4 -$$
$$z^3 \left[k_1 \left(u_0(x+1) + u_1 x + u_x\right) - k_2 \left(x\left(u_0 + u_1 + \theta_0 + \theta_1\right) + u_0 + \theta_0 + u_x + \theta_x\right)\right] +$$
$$O\left(z^2\right).$$

Setting

$$\det A_6 = R^{-2}(z)\left[k_1 k_2 z^4 + F_6 z^3 + O\left(z^2\right)\right] \tag{16}$$

we get the coefficient F_6 that determines the Whitham dynamics:

$$F_6 = \left(k_1 - k_2\right)\left(u_1 + x u_x\right) - x\left(2k_1 k_2 + \theta_x\right) - 2k_1 k_2 - k_2\theta_1. \tag{17}$$

The current goal is to extract the constraint on elliptic function from condition (17). To do this we use (14):

$$\theta_\infty\left(u_1 + x u_x\right) = -R(y)\widehat{u}^2 +$$
$$\widehat{u}\left[(y-1)(y-x)\left(2k_2 + \theta_\infty\right) - \theta_1(y-x) - x\theta_x(y-1)\right] +$$
$$k_1 k_2(x+1-y) + k_2\left(\theta_1 + x\theta_x\right),$$

which via (15) and (13) turns to the following:

$$\theta_\infty\left(u_1 + x u_x\right) = -\frac{x^2(x-1)^2}{4R(y)}\left(y'\right)^2 +$$
$$\frac{1}{2}y'x(x-1)\left\{B + \frac{1}{R(y)}\left[(y-1)(y-x)\left(2k_2 + \theta_\infty\right) - \theta_1(y-x) - x\theta_x(y-1)\right]\right\} -$$
$$\frac{1}{4}R(y)B^2 + \frac{1}{2}B\left[x\theta_x(y-1) - (y-1)(y-x)\left(2k_2 + \theta_\infty\right) + \theta_1(y-x)\right] +$$
$$k_1 k_2(x-y+1) + k_2\left(\theta_1 + x\theta_x\right),$$

where

$$B = \frac{\theta_0}{y} + \frac{\theta_1}{y-1} + \frac{\theta_x + 1}{y-x}.$$

Now substitute this into (17) and obtain final constraint on genus one Riemann surface (y', y) and appropriate elliptic uniformization (consider x and F_6 parameters):

$$\begin{gathered} x^2(x-1)^2\left(y'\right)^2 - 2y'x(x-1)y(y-1) + y^4\left[1-(k_1-k_2)^2\right] + \\ 2y^3\left[(k_1+k_2)C - 1 + 2x\theta_x(1-k_2) + 2F_6\right] - \\ y^2 S + \\ 2yx\left[2k_1k_2(x+1) + 2x\theta_x(1-k_2) + 2F_6 - \theta_0 C\right] - x^2\theta_0^2 = 0, \end{gathered} \tag{18}$$

where $C = (x+1)(k_1+k_2) + x\theta_x + \theta_1$,

$S = C^2 - 1 - 2x\theta_0(k_1+k_2) + 4k_1k_2\left(x^2+x+1\right) + 4x(x+1)(1-k_2)\theta_x + 4(x+1)F_6.$

To start the Whitham asymptotic analysis we need also the modulation equation in addition to ansatz (18). It can be found in the following way. First replace variables x to X in formula (16) and differentiate it in X :

$$D_X \det A_6 = \frac{z^4}{(z-X)R^2(z)}\left(2k_1k_2 + D_X F_6\right) + O\left(z^3\right). \tag{19}$$

On the other hand we have condition (8) which is to be studied now. Thus we have the following:

$$D_z L_6(z,x) = \frac{A_x}{(z-X)^2}, \quad D_z l_{22} = -\frac{u_x}{(z-X)^2},$$

whence obtain:

$$a_{11} D_z l_{22} = \frac{u_x}{R(z)(z-X)^2}\left[z^2 k_1 + O(z)\right],$$

$$a_{22} D_z l_{11} = -\frac{u_x + \theta_x}{R(z)(z-X)^2}\left[z^2 k_2 + O(z)\right],$$

substitute into (8) and get:

$$D_X \det A_6 = \frac{z^2\left[\overline{u}_x(k_1-k_2) - k_2\theta_x\right] + O(z)}{R(z)(z-X)^2}, \tag{20}$$

where bar means averaging. Comparison of formulas (19) and (20) yields the modulation equation:

$$D_X F_6 = \overline{u}_x\,(k_1 - k_2) - k_2\theta_x - 2k_1k_2. \tag{21}$$

One can as well rewrite (21) in the initial coordinates $y,\ X$. To do this use

$$u_x\,(k_1 - k_2) = \frac{y - X}{X(X-1)}S, \tag{22}$$

where

$$S = R(y)\widehat{u}^2 + \widehat{u}\,[\theta_1(y - X) + X\,(\theta_x + \theta_\infty)\,(y-1) - 2k_2(y-1)(y-X)] +$$
$$k_2^2(y-1) - k_2\,(\theta_1 + X\theta_x) - Xk_1k_2.$$

Now substitute (22) into (21), again utilize (13), (14), (15), simplify and finally obtain the modulation equation:

$$D_X F_6 = \frac{1}{2}\,(k_1 - k_2)\,D_X\overline{y} + \frac{(k_2 - k_1)\,(k_2 - k_1 + 1)}{2X(X-1)}\overline{y^2} + \tag{23}$$
$$\frac{\overline{y}}{X(X-1)}S +$$
$$\frac{1}{2(X-1)}\,[\theta_0\,(k_2 - k_1) + 2X\,(2k_1k_2 + \theta_x) + 2k_2\,(k_1 + k_2 + \theta_1) + 2F_6]$$
$$-k_2\theta_x - 2k_1k_2,$$

where

$$S = \frac{1}{2}\,(k_2 - k_1)\,[X\,(k_2 - k_1 - \theta_x) + \theta_0 + \theta_x + 1] -$$
$$X\,(2k_1k_2 + \theta_x) - k_2\,(k_1 + k_2 + \theta_1) - F_6.$$

Here $\overline{y}$ denotes the mean for elliptic function y specified by equation (18) where F_6 and X (instead of x) are considered as parameters.

3. Partial solutions for the modulation equation and PVI.

Analysis of the system (18), (23) in generic form is cumbersome, moreover there is a question of the phase shift Φ within the elliptic ansatz (5). This is

why we start with the simplest solutions that correspond to strongly degenerate surface (18). While trying to find partial solution to (18) - (23) among elementary functions one can note asymptotic homogeneity of formula (18) for large x. Denote $y = x\xi$ and rewrite (18) in variables $x\xi'$ and ξ. The discriminant for this polynomial looks as follows:

$$D = \xi^4 (k_2 - k_1)^2 - 2\xi^3 \left[(k_2 + k_1)^2 + 2\theta_x (1 - k_2) + 2F_6 x^{-1} + O\left(x^{-1}\right)\right] + \\ \xi^2 \left[(k_2 + k_1)^2 + 4k_1 k_2 + 4\theta_x (1 - k_2) + 4F_6 x^{-1} + O\left(x^{-1}\right)\right] - \\ 2\xi \left[2F_6 x^{-2} + O\left(x^{-2}\right)\right] + O\left(x^{-2}\right). \tag{24}$$

Seeking condition for strong degeneracy of the Riemann curve (18) one find out that the polynomial (24) tends to have two double roots if

$$\theta_x = 0 \;\; and \;\; F_6 = -2k_1 k_2 X + o(X), \quad X \to \infty, \tag{25}$$

$$D = \xi^2 (\xi - 1)^2 (k_2 - k_1)^2 + O\left(X^{-1}\right).$$

In this case four branch points of the Riemann surface (18) asymptotically coinside pairwise (that is what we call double or strong degeneracy). Substituting condition (25) into (18) one easily obtains the appropriate asymptotics for solution: $\xi = 1 + o(1)$ and, therefore,

$$y = x + o(x), \quad x \to \infty, \tag{26}$$

where $o(x)$ denotes terms that grow not faster than $\log x$. One can also easily verify that solution (25), (26) suits the modulation equation (23).

So we have the following theorem.

Theorem 2. In the case of $\theta_x = 0$ ($\delta = 1/2$) the sixth Painlevé equation has a solution with asymptotics (26)[1].

To prove this one should note in addition to mentioned above that strong degeneracy of the elliptic ansatz (18) transforms the phase shift Φ in formula (5) into a shift in variable x which can be found via simple iterative procedure computing terms of the series (26).

[1] Such a solution for PVI under $\alpha = (2\mu - 1)/2$, $\beta = \gamma = 0$, $\delta = 1/2$ and half-integer μ was found in paper [4]

References

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